We can use Dirichlet characters and the orthogonality relations to restrict arithmetic functions to arithmetic progressions:

<u>Lemma</u> (Usefulness of Dirichlet characters)

Let $f \in \mathcal{A}$ an arithmetic function and $a \in \mathbb{Z}$ with $(a,q)=1$.

Then

$$\sum_{\substack{n \le X \\ n \equiv a (q)}} f(n) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \sum_{n \le X} f(n)\chi(n)$$

<u>Proof</u>:

$$\sum_{\substack{n \le X \\ n \equiv a (q)}} f(n) \overset{\bar{a}a \equiv 1 (q)}{=} \sum_{\substack{n \le X \\ \bar{a}n \equiv 1 (q)}} f(n)$$

$$= \sum_{n \le X} \left( \frac{f(n)}{\varphi(q)} \sum_{\chi \bmod q} \chi(\bar{a}n) \right)$$

$$= \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \chi(\bar{a}) \sum_{n \le X} f(n)\chi(n). \quad \square$$

# Analytic continuation of $\zeta(s)$ and $L(s,\chi)$

Recall that $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ defined for $Re(s) > 1$

and $\zeta$ is holomorphic on $Re(s) > 1$. We can extend definition of $\zeta(s)$ to larger region.

<u>Theorem</u>: $\zeta$ admits unique meromorphic continuation to $\{s : Re(s) > 0\}$ with a simple pole at $s = 1$. Moreover, $\operatorname{Res}_{s=1} \zeta(s) = 1$.

<u>Proof</u>: Let $s \in \mathbb{C}$ with $Re(s) > 1$. By Abel summation,

$$\sum_{n \le x} \frac{1}{n^s} = \sum_{\frac{1}{2} < n \le x} \frac{\mathbb{1}(n)}{n^s} = \frac{\lfloor x \rfloor}{x^s} + s \int_{\frac{1}{2}}^{x} \frac{\lfloor t \rfloor}{t^{s+1}}\, dt$$

$$= \frac{\lfloor x \rfloor}{x^s} + s \int_{1}^{x} \frac{(t - \{t\})}{t^{s+1}}\, dt$$

$$= \frac{\lfloor x \rfloor}{x^s} + s \int_{1}^{x} \frac{1}{t^s} - s \int_{1}^{x} \frac{\{t\}}{t^{s+1}}\, dt$$

integral converges as $x \to \infty$

$$= \underbrace{\frac{\lfloor x \rfloor}{x^s}}_{\xrightarrow{x\to\infty} 0} + \frac{s}{s-1} - \underbrace{\frac{x^{1-s}}{s-1}}_{\xrightarrow{x\to\infty} 0} - s \int_{1}^{x} \frac{\{t\}}{t^{s+1}}\, dt.$$

We let $X \to \infty$, thus we obtain for $Re(S) > 1$

$$\zeta(S) = 1 + \frac{1}{S-1} - S \int_1^\infty \frac{\{t\}}{t^{S+1}} dt. \quad (*)$$

→ the function $1 + \frac{1}{S-1}$ is meromorphic in entire complex plane, simple pole at $S = 1$, and residue 1.

→ for $Re(S) > 0$, the integral converges absolutely. Thus defines holomorphic function on $Re(S) > 0$.

Hence, RHS of $(*)$ defines analytic function on $Re(S) > 0$ with simple pole at $S = 1$, which coincides with $\sum_{n=1}^\infty n^{-s}$ on $Re(S) > 1$.

The holomorphic extension of $\zeta(S)$ to $\{S \in \mathbb{C} : Re(S) > 0\} - \{1\}$ is unique, as it completely determined on $\{Re(S) > 1\}$.

(B)

<u>Theorem</u> (Analytic continuation for $L(S; \chi_0)$).

Let $\chi_0$ be the principal character mod $q$.

Then $L(S, \chi_0)$ admits unique meromorphic continuation to $Re(S) > 0$, with a simple pole at $S = 1$

and residue $\operatorname{Res}_{s=1} L(s,\chi_0) = \frac{\varphi(q)}{q}$.

<u>Proof</u>: By Euler product, for $\operatorname{Re}(s) > 1$,

$$L(s,\chi_0) = \prod_p \left(1 - \frac{\chi_0(p)}{p^s}\right)^{-1} = \prod_{(p,q)=1} \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$= \zeta(s) \cdot \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right).$$

Note that $\prod_{p \mid q} \left(1 - \frac{1}{p^s}\right)$ defines a holomorphic function for all $s \in \mathbb{C}$ and that $\prod_{p \mid q} \left(1 - \frac{1}{p}\right) = \frac{\varphi(q)}{q}$.

Conclusion follows from meromorphic continuation of $\zeta(s)$ to $\operatorname{Re}(s) > 0$. ☒

<u>Theorem</u> (Analytic continuation for non-principal characters).

If $\chi$ mod $q$ <u>NOT</u> the principal character, then $\sigma_c(\chi) = 0$. In particular, $L(s,\chi)$ has holomorphic contiunuation to $\operatorname{Re}(s) > 0$.

(so no poles)

<u>Proof</u>: Note that $\sum_{y < n \leq y+q} \chi(n) = 0$, $\forall y > 0$.

Define $S_\chi(N) := \sum_{n \le x} \chi(n)$. Then $|S_\chi(n)| \le \varphi(q)$.

We need to show $\left|\sum_{y<n\le x} \frac{\chi(n)}{n^s}\right| < \varepsilon$, whenever

$M_\varepsilon < y < x$ and $\operatorname{Re}(s) > 0$.

By Abel summation,

$$\sum_{y<n\le x} \frac{\chi(n)}{n^s} = \frac{S_\chi(x)}{x^s} - \frac{S_\chi(y)}{y^s} + s\int_y^x \frac{S_\chi(t)}{t^{s+1}}\,dt$$

Hence $\left|\sum_{y<n\le x} \frac{\chi(n)}{n^s}\right| < \varphi(q)\left(\frac{1}{x^\sigma} + \frac{1}{y^\sigma}\right) + |s|\varphi(q)\int_y^x \frac{1}{t^{\sigma+1}}\,dt$

$$= \varphi(q)(x^{-\sigma} + y^{-\sigma}) + \frac{|s|\varphi(q)}{\sigma}(y^{-\sigma} - x^{-\sigma}) \longrightarrow 0$$

as $y \to \infty$.

(if $\sigma > 0$).

We note $\sum_{n\le x} \frac{\chi(n)}{n^s}$ convergent for $\operatorname{Re}(s) > 0$

$\Rightarrow \sigma_c(\chi) = 0$

$\Rightarrow L(s,\chi)$ holomorphic for $\operatorname{Re}(s) > 0$. $\blacksquare$

**Theorem** (Dirichlet's theorem on primes in arithmetic progressions)

Let $q, a$ coprime integers.

Then $\{p \in \mathbb{P} : p \equiv a \bmod q\}$ is infinite.

**Motivation for proof**: There are infinitely many primes.

For $\sigma > 1$, we have $\zeta(\sigma) = \prod_p \left(1 - \frac{1}{p^\sigma}\right)^{-1}$

$$\Rightarrow \log \zeta(\sigma) = -\sum_p \log\left(1 - \frac{1}{p^\sigma}\right) = \sum_p \frac{1}{p^\sigma} + O(1)$$

(use Taylor expansion $\log(1-y)$)

But we know $\lim_{\sigma \downarrow 1} \zeta(\sigma) = \infty$

$$\Rightarrow \lim_{\sigma \downarrow 1} \sum_p \frac{1}{p^\sigma} = \infty$$

$\Rightarrow \sum_p \frac{1}{p} = \infty \Rightarrow$ there are infinitely many primes.

(proof is direct consequence of Euler product).

**Proof preparation for Dirichlet's theorem**:

We will show $\sum_{p \equiv a (q)} \frac{1}{p}$ is infinite.

By orthogonality relations,

$$\sum_{\substack{p \le x \\ p \equiv a (q)}} \frac{1}{p^s} = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi(a)} \underbrace{\sum_{p \le x} \frac{\chi(p)}{p^s}}_{=: S_\chi(x)}$$

Recall for $Re(s) > 1$, $L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$.

$$\Rightarrow \log(L(s,\chi)) = -\sum_p \log\left(1 - \frac{\chi(p)}{p^s}\right) + 2\pi i m,$$ for some $m \in \mathbb{Z}$.

Let $s \to \infty$ (along $\mathbb{R}$), we have $L(s,\chi) \to 1$, and similarly for RHS

$\Rightarrow m = 0$.

$$\log(L(s,\chi)) = -\sum_p \log\left(1 - \frac{\chi(p)}{p^s}\right) = \underbrace{\sum_p \frac{\chi(p)}{p^s}}_{= \lim_{x\to\infty} S_\chi(x)} + O(1)$$

For $Re(s) > 1$,

$$\Rightarrow \sum_{p \equiv a (q)} \frac{1}{p^s} = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \log L(s,\chi) + O(1). \quad (*)$$

$\rightarrow$ need to make sense of $L(s,\chi)$ at $s = 1$.

Want to take limit as $\sigma \searrow 1$.
We know $\lim_{\sigma \downarrow 1} \log(L(\sigma, \chi_0)) = \infty$ and want

to show the other contribution are finite. Hence Dirichlet's theorem follows from:

<u>Theorem</u>: (Dirichlet) Let $\chi \neq \chi_0$. Then $L(1, \chi) \neq 0$.

<u>Corollary</u>: $\lim_{\sigma \downarrow 1} \sum_{p \equiv a(q)} \frac{1}{p^\sigma} = \infty$.

<u>Proof</u>: We note that for $\sigma > 1$,

$$\sum_{\chi \bmod q} \log(L(\sigma, \chi)) = \sum_{\chi \bmod q} \sum_{p} \sum_{l=1}^{\infty} \frac{\chi(p)^l}{l p^{l\sigma}}$$

$$= \sum_{p} \sum_{l=1}^{\infty} \frac{1}{l p^{l\sigma}} \sum_{\chi \bmod q} \chi(p^l) \quad \text{(from absolute convergence)}$$

$$= \sum_{p} \sum_{l : p^l \equiv 1 (q)} \varphi(q) \frac{1}{l p^{l\sigma}} > 0.$$

$\Rightarrow$ for $\sigma > 1$, $\prod_{\chi \bmod q} L(\sigma, \chi) > 1$.

Recall for $\chi \neq \chi_0$, $L(s,\chi)$ holomorphic on $Re(s)>0$,
so $L(s,\chi) = (s-1)^{h_\chi} \cdot M_\chi(s)$,
where $h_\chi \in \mathbb{N} \cup \{0\}$ (order of the zero at $1$).
$M_\chi(s)$ holomorphic on $Re(s)>0$, $M_\chi(1) \neq 0$.
For $\chi_0$, $L(s,\chi_0)$ has simple pole at $s=1$.

$$\Rightarrow \prod_{\chi \bmod q} L(s,\chi) = M(s) \cdot (s-1)^{-1+\sum_{\chi\neq\chi_0} h_\chi},$$

where $M$ holomorphic on $Re(s)>0$.
Since $\{\prod_{\chi \bmod q} L(\sigma,\chi) : \sigma > 1\}$ is bounded away from $0$,

we must have $\displaystyle\sum_{\chi\neq\chi_0} h_\chi \leq 1$

(there is at most one $\chi$ mod $q$ s.t. $L(1,\chi)=0$).

<u>**Step 1**</u>: Non-vanishing for complex characters.
We show $h_\chi = h_{\overline{\chi}}$.
Indeed, $L(s,\overline{\chi}) = \displaystyle\sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^s} = \overline{\sum_{n=1}^{\infty} \frac{\chi(n)}{n^{\overline{s}}}}$ $\quad (\overline{n^s} = n^{\overline{s}})$

$$= \overline{L(\overline{s},\chi)} \quad \text{for } Re(s)>1.$$

From meromorphic continuation, identity holds also for $Re(s)>0$.

Hence $L(1,\chi)=0 \Rightarrow L(1,\bar{\chi})=0$.
So if $\chi \neq \bar{\chi}$, we must have $h_\chi = h_{\bar{\chi}} = 0$.

## Step 2: Non-vanishing for non-principal real (quadratic) characters.

Suppose $\chi = \bar{\chi}$, i.e. $\chi^2 = \chi_0$ $(\chi(n) \in \{0, \pm 1\})$
but $\chi \neq \chi_0$.

Define $r(n) = \sum_{d|n} \chi(d) = \chi * \mathcal{E}(n)$
$\rightarrow r$ multiplicative

We have that $r(p^l) = \begin{cases} 1, & \text{if } p \mid q \\ l+1, & \text{if } (p,q)=1 \text{ and } \chi(p)=1 \\ 1, & \text{if } \chi(p)=-1 \text{ and } l \text{ even} \\ 0, & \text{if } \chi(p)=-1 \text{ and } l \text{ odd} \end{cases}$

(exercise)

In particular, $r(n) \geq 0$ and $r(n^2) \geq 1$, $\forall n \in \mathbb{N}$.

Suppose for contradiction $L(1,\chi)=0$.
Then $L_r(s) = L_\chi(s)\, \zeta(s)$ holomorphic
for $\{\operatorname{Re} s > 0\}$.

[since $\zeta(s) = \frac{1}{s-1} M_\zeta(s)$, $L_\chi(s) = (s-1)^{h_\chi} M_\chi(s)$ $M_\zeta(s)$ & $M_\chi(s)$ holomorphic on $\Re(s)>0$

$\Rightarrow L_r(s) = (s-1)^{h_\chi - 1} M_\chi(s) M_\zeta(s)$ ]

**Theorem** (Landau)

Let $f \in \mathcal{A}$ with $\sigma_c(f) < \infty$ and $f(n) \geq 0$, $\forall n \in \mathbb{N}$. Then $L_f(s)$ has singularity (i.e. not holomorphic) at $s = \sigma_c(f)$.

Assuming Landau, it follows $\sigma_c(r) \leq 0$.

But $\sum_{n \leq X} \frac{r(n)}{\sqrt{n}} \geq \sum_{n \leq \sqrt{X}} \frac{r(n^2)}{n} \geq \sum_{n \leq \sqrt{X}} \frac{1}{n} = \frac{1}{2} \lg x + O(1)$

$r(n) \geq 0$ $\qquad r(n^2) \geq 1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{r(n)}{n^{1/2}}$ diverges, contradiction.

This concludes proof of Dirichlet's theorem.

$\square$

Remark: From the proof, it follows that if $\chi$ non-principal quadratic character and $\sigma > 1$, $\zeta(\sigma) L_\chi(\sigma) = \sum_n \frac{r(n)}{n^\sigma} > 0$.

As $\zeta(\sigma) > 0 \Rightarrow L(1, \chi) > 0$.